

UNIFORMLY RATIONAL VARIETIES WITH TORUS ACTION

ALVARO LIENDO AND CHARLIE PETITJEAN

ABSTRACT. A smooth variety is called uniformly rational if every point admits a Zariski open neighborhood isomorphic to a Zariski open subset of the affine space. In this note we show that every smooth and rational affine variety endowed with an algebraic torus action such that the algebraic quotient has dimension 0 or 1 is uniformly rational.

INTRODUCTION

In the seminal paper [4], Gromov discusses the class of algebraic varieties X having the property that every point admits a Zariski open neighborhood isomorphic to a Zariski open subset of the affine space. These varieties are now called uniformly rational varieties [2]. A uniformly rational variety is clearly smooth and rational. Furthermore, in dimensions 1 and 2, every smooth and rational variety is uniformly rational. In higher dimension, it is an open question whether all smooth and rational varieties are uniformly rational.

Several families of uniformly rational varieties are known. For instance, rational varieties that are homogeneous with respect to the action of an algebraic group are uniformly rational. Smooth toric varieties are also uniformly rational. Furthermore, blow-ups along smooth centers of uniformly rational varieties are uniformly rational.

In this note we deal with smooth varieties endowed with torus actions that are not necessarily toric. We work over the field of complex numbers \mathbb{C} . Let T be the n -dimensional algebraic torus \mathbb{G}_m^n , where \mathbb{G}_m is the multiplicative group. The complexity of a T -action is the codimension of a generic orbit. Since the quotient of an algebraic torus by a normal algebraic subgroup is again an algebraic torus, up to changing the torus, we can always assume that a T -action is faithful. A T -variety is a normal variety endowed with a faithful T -action. Hence, toric varieties correspond to T -varieties of complexity zero.

As stated before, all smooth toric varieties are uniformly rational. This follows from the fact that every smooth toric variety without torus factor admits a T -equivariant open cover by affine charts isomorphic to the affine space. Furthermore, it follows in a straightforward way from [6, Chapter 4] that smooth and rational T -varieties of complexity one are also uniformly rational. Moreover, if X is also complete then X also admits a T -equivariant open cover by affine charts isomorphic to the affine space, see [1].

In higher complexity the situation is less clear. It is not known whether all smooth and rational T -varieties of complexity two or higher are uniformly rational, but the second author has provided counterexamples to an equivariant version of uniform rationality. A G -variety X is said to be equivariantly uniformly rational if it admits an G -invariant open cover by open sets G -equivariantly isomorphic to G -invariant open sets of the affine space endowed with a G -action. In [7], the author has given examples of affine T -varieties of complexity two and higher that are not equivariantly uniformly rational.

Nevertheless, we prove that a large class of affine T -varieties are uniformly rational. Indeed, we show that a smooth and rational affine T -variety X of any complexity is uniformly rational,

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provided that the algebraic quotient is of dimension at most one. Recall that the algebraic quotient $X//T := \text{Spec}(\mathbb{C}[X]^T)$ is the spectrum of the ring of invariant regular function. More precisely, we will prove the following theorem.

Theorem. *Let X be a smooth and rational affine T -variety.*

- (1) *If the algebraic quotient $X//T$ is a point then X is equivariantly isomorphic to $(\mathbb{C}^*)^l \times \mathbb{A}^{n-l}$. In particular, X is uniformly rational.*
- (2) *If the algebraic quotient $X//T$ is a curve then X is uniformly rational.*

PROOF OF THE RESULT

Let $N \simeq \mathbb{Z}^k$ be a lattice of rank k , and let $M = \text{Hom}(N, \mathbb{Z})$ be its dual lattice. We let $T = \text{Spec}(\mathbb{C}[M])$ be the algebraic torus of dimension k . This ensures that M is character lattice of T and N is the 1-parameter subgroup lattice of T . We also let $M_{\mathbb{Q}}$ be the \mathbb{Q} -vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_{\mathbb{Q}}$ be the \mathbb{Q} -vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a natural duality pairing $\langle \cdot, \cdot \rangle : M_{\mathbb{Q}} \times N_{\mathbb{Q}} \rightarrow \mathbb{Q}$.

We consider now an algebraic affine T -variety $X = \text{Spec } A$. The T -action on X corresponds to an M -grading of the ring A of regular functions where for every $u \in M$, the homogeneous piece $A_u \subset A$ is given by the semi-invariant functions with respect to the character χ^u , i.e.,

$$A = \bigoplus_{u \in M} A_u \quad \text{with} \quad A_u = \{f \in A \mid \lambda \cdot f = \chi^u(\lambda) \cdot f\}.$$

The weight monoid S attached to the T -action on X corresponds to all element $u \in M$ such that $A_u \neq \{0\}$. The cone spanned by S in the vector space $M_{\mathbb{Q}}$ is called the weight cone of the T -action and we denote it by ω .

Our proof of the theorem is a consequence of a canonical factorization of the quotient morphism $\pi : X \rightarrow X//T = \text{Spec } A_0$ that may be of independent interest. We state this factorization in the following proposition. First, we need some definitions.

A T -action is said to be fix-pointed if the only vector space contained in the weight cone ω is $\{0\}$. This is the case if and only if algebraic quotient $X//T$ is isomorphic to the fixed point locus X^T via the composition $X^T \hookrightarrow X \xrightarrow{\pi} X//T$. A T -action is said to be hyperbolic if the weight cone ω is the whole $M_{\mathbb{Q}}$. This is the case if and only if the dimension of the algebraic quotient coincides with the complexity of the T -action.

Proposition. *For every T -variety X there is a unique splitting of the acting torus $T = T_1 \times T_2$ such that*

- (1) *The action of T_1 on X is fix-pointed.*
- (2) *The torus T_2 acts (non necessarily faithfully) on $X_H := X//T_1$ and this action is hyperbolic.*
- (3) *The quotient morphism $\pi : X \rightarrow X//T$ factorizes as*

$$X \xrightarrow{//T_1} X_H \xrightarrow{//T_2} X_H//T_2 \simeq X//T.$$

Proof. Let H be the biggest vector space contained in $\omega \subseteq M_{\mathbb{Q}}$. We define the finitely generated and graded algebra $A_H = \bigoplus_{m \in H \cap M} A_m$ which gives us the algebraic variety $X_H := \text{Spec } A_H$. This yields a sequence of inclusion of algebras

$$A \hookrightarrow A_H \hookrightarrow A_0,$$

where A_0 is the ring of invariant functions for the T -action and so is the algebra of functions of the algebraic quotient $X//T$.

Let $N_1 = H^{\perp} \cap N$ be the sublattice of N of vectors orthogonal to H . By definition, N_1 is saturated as sublattice. Then the quotient N/N_1 is torsion free which by [3, Exercice 1.3.5.] implies the existence of a complementary sublattice $N_2 \subseteq N$ such that $N = N_1 \oplus N_2$. Let M_i

be the dual lattice of N_i and $T_i = \text{Spec } \mathbb{C}[M_i]$ the associated algebraic torus for $i = 1, 2$. This yields a splitting $T = T_1 \times T_2$ of the original torus.

The previous sequence of inclusion of algebra provides the following sequence of algebraic quotients

$$X \xrightarrow{//T_1} X_H \xrightarrow{//T_2} X_H // T_2 \simeq X // T.$$

By construction, the action of T_1 on X is fix-pointed and faithful whereas the action of T_1 on X_H is hyperbolic. The uniqueness of such splitting is clear from the construction. \square

In the proof of our theorem we need the following result directly borrowed from [5, Theorem 2.5].

Lemma. *With the above notation, if X is smooth then X_H is smooth and X admits a structure of a vector bundle over X_H where each fiber of the vector bundle is stable under the T_1 -action and T_1 acts linearly on it.*

In [5] a fix-pointed action is called unmixed. We prefer to call it fix-pointed since it is a notion that makes sense for algebraic groups different from the torus. Remark that part (1) of our theorem provides a slight generalization of the above lemma in the case where $X // T$ is reduced to a point.

Proof of the theorem. To prove (1), remark that X_H admits a hyperbolic T_2 -action and by hypothesis we have $\dim(X // T) = \dim(X_H // T_2) = 0$. Since for a hyperbolic action the dimension of the algebraic quotient equals the complexity we have that the T_2 -action is of complexity zero and so X_H is a toric variety.

Since X is smooth, we have that X_H is smooth and X has the structure of a vector bundle over X_H by the lemma. Hence X_H is isomorphic to $(\mathbb{C}^*)^a \times \mathbb{A}^b$. By the generalization of the Quillen-Suslin theorem [8, 9] (see [10] for the generalization) any vector bundle over a ring of Laurent polynomial is trivial. Thus X is a smooth toric variety isomorphic to $(\mathbb{C}^*)^l \times \mathbb{A}^{n-l}$.

Let us now prove the second assertion of the theorem. Since X is rational, we have that $X // T$ is unirational. By hypothesis, $X // T$ is a curve so we obtain that $X // T$ is a rational curve. This shows that X_H is also a rational variety.

Furthermore, by the lemma we have that X_H is smooth. Since X_H admits an hyperbolic T -action and by hypothesis $\dim(X // T) = \dim(X_H // T_1) = 1$, with the same argument above we obtain that the T_2 -action on X_H is of complexity one. Then X_H is uniformly rational by [6, Chapter 4]. Finally, by the lemma X is a vector bundle over X_H and any vector bundle over a uniformly rational variety is uniformly rational (see also [2, Example 2.1]). \square

Example. Let X be the hypersurface in \mathbb{A}^5 given by

$$X = \{(x, y, z, t, u) \in \mathbb{A}^5 \mid zty + x^2 + y + t^2u = 0\}.$$

Let $T = \mathbb{G}_m^2$. The variety is T -stable for the linear T -action on \mathbb{A}^5 given by

$$(\lambda_1, \lambda_2) \cdot (x, y, z, t, u) = (\lambda_1 \lambda_2 x, \lambda_1^2 \lambda_2^2 y, \lambda_1 z, \lambda_1^{-1} t, \lambda_1^4 \lambda_2^2 u).$$

Then X is a smooth and rational T -variety of complexity two. The algebraic quotient $X // T$ is $\mathbb{A}^1 = \text{Spec } \mathbb{C}[zt]$. Hence, X is uniformly rational by our Theorem.

Remark. In the situation where $\dim(X // T) \geq 2$ it was proven by the second author in [7] that there exist smooth and rational affine threefolds endowed with an hyperbolic \mathbb{G}_m -action that are not \mathbb{G}_m -uniformly rational. For instance the hypersurface X in \mathbb{A}^4 given by

$$\{(x, y, z, t) \in \mathbb{A}^4 \mid z^2 y + x^3 y^2 + x + t^3 = 0\}.$$

admits a \mathbb{G}_m -action obtained by restricting the linear \mathbb{G}_m -action on \mathbb{A}^4 given by

$$\lambda \cdot (x, y, z, t) = (\lambda^3 x, \lambda^{-3} y, \lambda^3 z, \lambda t).$$

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INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CASILLA 721, TALCA, CHILE.

E-mail address: aliendo@inst-mat.otalca.cl

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CASILLA 721, TALCA, CHILE.

E-mail address: petitjean.charlie@gmail.com